

# On the shelling antimatroids of split graphs

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## Abstract

Unlike poset antimatroids, chordal graph shelling antimatroids have received little attention as regard their structures, optimization properties and associated circuits. Here we consider a special case of those antimatroids, namely the split graph shelling antimatroids. We establish a connection between the structure of split graph shelling antimatroids and poset shelling antimatroids. We discuss some applications of this new connection, in particular, we give a simple polynomial time algorithm to find a maximum (or minimum) weight feasible set in split graph shelling antimatroids and list all the circuits and free sets for this class of antimatroids.

## 1 Introduction

The “split graph shelling antimatroids” are both generalizations of some “poset antimatroids” and particular instances of “chordal graph shelling antimatroids”. All the terms are explained in the next subsections. Our results shed light on the structure of split graph shelling antimatroids. We infer from them a polynomial time algorithm to find an optimal feasible set in a split graph shelling antimatroid whose elements are weighted.

### Antimatroids

Antimatroids arise naturally from various kinds of shellings and searches on combinatorial objects, and appear in various contexts in mathematics and computer science. Dilworth [5] first examined structures very close to antimatroids in terms of lattice theory. Later, Edelman [6] and Jamison [15] studied the convex aspects of antimatroids. Korte, Lovász and Schrader [18] considered antimatroids as a subclass of greedoids. Today, the concept of antimatroid appears in many fields of mathematics such as formal language theory (Boyd and Faigle [3]), choice theory (Koshevoy [19]), game theory (Algaba *et al.* [1]) and mathematical psychology (Falmagne and Doignon [10]) among others. The concept of a convex geometry is dual to the one of an antimatroid.

A set system  $(V, \mathcal{F})$ , where  $V$  is a finite set of elements and  $\emptyset \neq \mathcal{F} \subseteq 2^V$ , is an *antimatroid* when

$$V \in \mathcal{F}, \tag{AM0}$$

$$\forall F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cup F_2 \in \mathcal{F}, \tag{AM1}$$

$$\forall F \in \mathcal{F} \text{ and } F \neq \emptyset \Rightarrow \exists f \in F \text{ such that } F \setminus \{f\} \in \mathcal{F}. \tag{AM2}$$

The condition (AM2) is called the *accessibility property*. The *feasible sets* of the antimatroid  $(V, \mathcal{F})$  are the members of  $\mathcal{F}$ . The *convex sets* are the complements in  $V$  of the feasible sets.

### Shellings

Antimatroids also relate to special shelling processes. Given a feasible set  $F$  in an antimatroid  $(V, \mathcal{F})$ , a *shelling* of  $F$  is an enumeration  $f_1, f_2, \dots, f_{|F|}$  of its elements such that  $\{f_1, f_2, \dots, f_k\}$  is feasible for any  $k$  with  $1 \leq k \leq |F|$ . In view of the accessibility property, any feasible set admits at least one shelling. There is an axiomatization of antimatroids in terms of shellings (see

for example [18]). Many examples of antimatroids arise in a natural way from shelling processes. The next two subsections present the cases of posets and of chordal graphs.

## Poset antimatroids

Recall that a *poset*  $P$  is a pair  $(V, \leq)$  formed of a finite set  $V$  and a binary relation  $\leq$  over  $V$  which is reflexive, antisymmetric, and transitive. For a poset  $(V, \leq)$  a *filter*  $F$  is a subset of  $V$  such that for all elements  $a$  in  $F$  and  $b$  in  $V$ , if  $a \leq b$ , then  $b$  is also in  $F$ . The filters are also known as *upper ideals* or *ending sets*. We denote the family of all filters of  $P$  as  $\text{flt}(V, \leq)$ .

One particular class of antimatroids comes from shelling processes over posets by removing successively the maximum elements. Let  $(V, \leq)$  be a poset, then  $(V, \text{flt}(V, \leq))$  is a *poset (shelling) antimatroid*. Thus the feasible elements are the filters. The class of poset antimatroids is often considered as one of the most basic, because it arises in many different contexts. Poset antimatroids are the only antimatroids closed under intersection [18]. There exist several other characterizations for this class of antimatroids. Nakamura obtains a characterization of poset antimatroids by single-element extensions in [21] and by excluded minors in [22]. Recently, Kempner and Levit [17] introduce the poly-dimension of an antimatroid, prove that every antimatroid of poly-dimension 2 is a poset antimatroid, and establish both graph and geometric characterizations of such antimatroids.

From the optimization point of view, poset antimatroids with weighted elements are easy to study. First, because Picard [24] made a direct connection between finding a maximum (or minimum) weight feasible set in a poset antimatroid and a maximum flow problem. Second, Stanley [25] provides a complete linear description of the convex hull of the characteristic vectors of the feasible sets.

## Chordal graph shelling antimatroids

Another particular class of antimatroids comes from shelling processes over *chordal graphs*, *i.e.* graphs in which every induced cycle in the graph has at most three vertices. For a background on chordal graphs, see Golumbic [13]. Any chordal graph has at least one *simplicial* vertex, *i.e.* a vertex such that its neighbors induce a clique. For any chordal graph  $G = (V, E)$ , we define an antimatroid  $(V, \mathcal{F})$  in which a subset  $F$  of  $V$  is feasible if and only if there is some ordering  $O = (f_1, \dots, f_{|F|})$  of the elements of  $F$  such that for all  $j$  between 1 and  $|F|$ ,  $f_j$  is simplicial in  $G \setminus \{f_1, \dots, f_{j-1}\}$ . The antimatroid resulting from this construction is called a *chordal graph (vertex) shelling antimatroid*. The ordering  $O$  is called a *simplicial shelling* of  $F$ . In fact, only the chordal graphs form an antimatroid based on their simplicial shellings (Farber and Jamison [11]).

We recall that a *split graph* is a graph whose set of vertices can be partitioned into a clique and an independent set. Here we assume that for every split graph, the partition is given and we will denote by  $K$  and  $I$  respectively, the clique and the independent set. Split graphs are chordal graphs, and they are the only chordal graphs to be co-chordal (*i.e.* the complement of the graph is also chordal), see Golumbic [13]. Here we consider the special case of chordal graph shelling antimatroids where the graph is a split graph. These antimatroids will be called *split graph (vertex) shelling antimatroids*. Split graphs are relatively well known and have a wide range of theoretical use, see for instance [20, 13].

## Structure of the paper

The aim of this work is to provide a better understanding of the split graph shelling antimatroids. In Section 2, we establish a new characterization of the feasible sets of split graph shelling antimatroids

and discuss the connection with the poset antimatroids. In Section 3 we prove a hardness result about optimization problems on antimatroids and develop a polynomial time algorithm to find a maximum (or minimum) weight feasible set in split graph shelling antimatroids. Finally in Section 4 we use the previous results to list all the circuits and free sets of the split graph shelling antimatroids. This work is a first step to a better understanding of a more general class: the chordal graph shelling antimatroids.

## Notation

Let  $V$  denote a set of elements. For  $S \subseteq V$ , the *complement* of  $S$  is  $S^c = V \setminus S$ . The set of all subsets of  $V$  is denoted as  $2^V$  and the symbol  $\uplus$  is used for the disjoint union of two sets. Let  $G = (V, E)$  be a simple graph; we write  $u \sim v$  as a shortcut for  $\{u, v\} \in E$ , and  $u \not\sim v$  for  $\{u, v\} \notin E$ . For  $V' \subseteq V$  we denote by  $N(V')$  the set of vertices *adjacent* to  $V'$ , *i.e.* the vertices  $w$  in  $V \setminus V'$  such that  $w \sim v$  for some  $v$  in  $V'$ . We write  $N(v)$  for  $N(\{v\})$ . We call a vertex *isolated* if  $N(v) = \emptyset$ .

## 2 The split graph shelling antimatroids

### Characterization of the feasible sets

Here is a useful characterization of the feasible sets in a split graph shelling antimatroid. Example 1 provides an illustration.

**Proposition 2.1.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ . Then a subset  $F$  of vertices is feasible for the antimatroid if and only if  $N(F)$  induces a clique.*

*Proof.* For the necessary condition (Fig. 1), suppose we have a simplicial shelling  $O = (f_1, \dots, f_{|F|})$  of a feasible set  $F$  such that  $N(F)$  does not induce a clique in  $G$ . Then, for some vertices  $v_1$  and  $v_2$  in  $N(F)$  we have  $v_1 \not\sim v_2$ . Hence  $\{v_1, v_2\} \not\subseteq K$ , since  $K$  is a clique. Assume without loss of generality that  $v_1 \in I$ . Let  $f_j$  be the first element in  $O$  such that  $f_j \sim v_1$ . As  $v_1 \in I$ , by definition of a split graph,  $f_j \in K$  and  $f_j$  is adjacent to all other vertices of  $K$ . Then  $f_j$  is not adjacent to  $v_2$  because  $f_j$  must be simplicial in  $G \setminus \{f_1, \dots, f_{j-1}\}$ , so  $v_2$  must be in  $I$ . Now let  $f_t$  be the first element of  $O$  such that  $f_t \sim v_2$  (notice  $j \neq t$ ). Since  $v_2 \in I$ , by a completely symmetric argument, we have  $f_t \in K$  and  $f_t \sim v_1$ . Now a contradiction follows because, if  $j > t$ , the vertex  $f_j$  is not simplicial in  $G \setminus \{f_1, \dots, f_{j-1}\}$ , and if  $t > j$  the vertex  $f_t$  is not simplicial in  $G \setminus \{f_1, \dots, f_{t-1}\}$ .

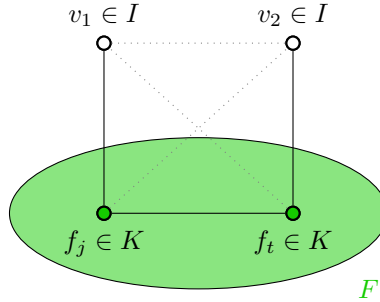


Figure 1: Illustration of the proof of necessary condition for Proposition 2.1

Reciprocally, suppose that we have a set of vertices  $F$  such that  $N(F)$  induces a clique (Fig. 2). We will build a simplicial shelling  $O$  on  $F$  with the help of the following three set partition of  $F$ :

$$\begin{aligned} V_1 &= F \cap I, \\ V_2 &= (F \cap K) \setminus N(I \setminus F), \\ V_3 &= (F \cap K) \cap N(I \setminus F). \end{aligned}$$

We arbitrarily order the elements in each of the sets  $V_1, V_2, V_3$  and concatenate the orderings in this order to obtain the sequence  $O$ . By the definition of a split graph, it is obvious that the elements of  $O$  in  $V_1 \cup V_2$  fulfill the condition of a simplicial shelling. If  $V_3 = \emptyset$ , we are done. Otherwise,  $N(V_3) \setminus F$  is a clique and so it has exactly one element  $i$  in  $I$ , because  $N(F)$  induces a clique thus all elements of  $V_3$  are adjacent to this single element of  $I \setminus F$ . Therefore the elements of  $O$  in  $V_3$  fulfill the conditions of the simplicial shelling.  $\square$

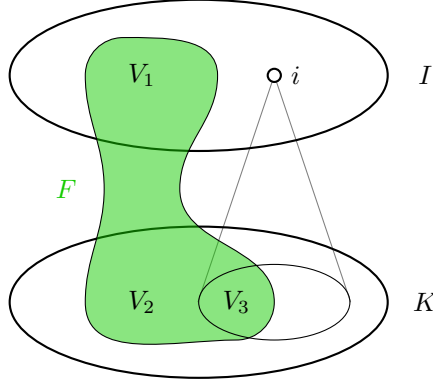


Figure 2: Illustration of the proof of the sufficient condition for Proposition 2.1

**Example 1.** Figure 3 below shows two split graphs on which we build a split graph shelling antimatroid. The set  $F$  on the left (Fig. 3a) is a feasible set and we see that  $N(F)$  defines a clique. On the right (Fig. 3b), we have a clique  $C$  and a possible simplicial shelling is proposed for a set of vertices such that its neighborhood is  $C$ .

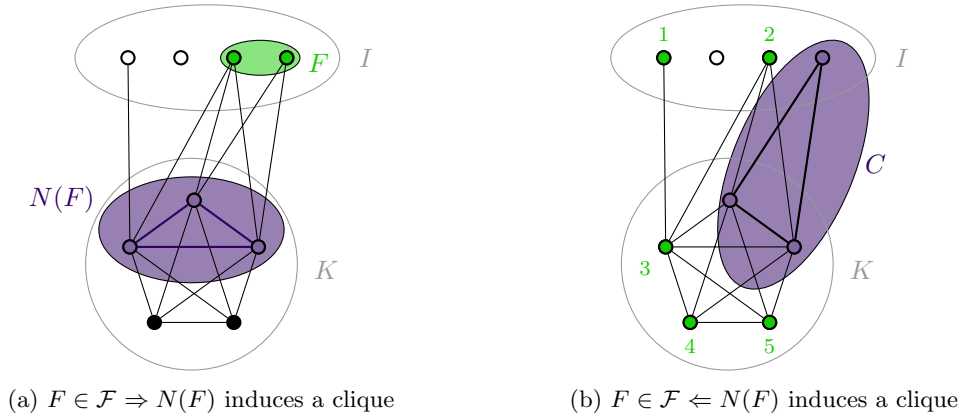


Figure 3: Examples for Proposition 2.1

In a graph  $(V, E)$ , a subset  $C$  of  $V$  is *monophonically convex* (*m-convex*) when  $C$  contains all the vertices of all chordless paths joining any two vertices of  $C$ .

**Proposition 2.2.** *Let  $(V, E)$  be a graph and  $F$  be a subset of  $V$ . If  $N(F)$  is a clique, then  $V \setminus F$  is  $m$ -convex.*

*Proof.* Assume  $N(F)$  is a clique. Proceeding by contradiction, we take two vertices  $v, w$  in  $V \setminus F$  for which there exists a chordless path  $v = u_0, u_1, \dots, u_k = w$  having at least one vertex in  $F$ ; here we may moreover chose  $v$  and  $w$  in order that  $k$  be as small as possible. Then necessarily  $u_1$  and  $u_{k-1}$  are in  $F$  (with possibly  $u_1 = u_{k-1}$ ). By the assumption on  $N(F)$ ,  $v$  and  $w$  must be adjacent and thus they form a chord of the path, a contradiction.  $\square$

The converse of the implication in Proposition 2.2 does not hold even if the graph is connected. Even more:  $V \setminus F$  being  $m$ -convex does not imply that the graph  $N$  induced on  $N(F)$  is a parallel sum of cliques (in other words, that  $N$  is the complement of a multipartite graph). Figure 4 below shows a counter-example based on a 2-connected, chordal graph.

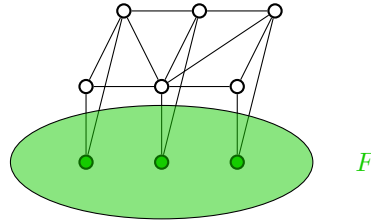


Figure 4: Counter-example for the converse implication in Proposition 2.2

For a split graph  $(V, E)$ , the converse of the implication in Proposition 2.2 also holds (this follows from Proposition 2.1 and Section 3 of [11]).

**Corollary 2.1.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ . For all feasible sets  $F$ , there exists at most one  $i \in I \setminus F$  such that there is  $k \in K \cap F$  with  $k \sim i$ .*

*Proof.* This comes directly from Proposition 2.1 and the fact that the set  $I$  is an independent set.  $\square$

**Corollary 2.2.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ . Let  $u$  and  $v$  be in  $V$ . Then  $V \setminus \{u, v\} \notin \mathcal{F}$  implies  $u \sim v$  in  $G$ . Moreover if  $V \setminus \{u, v\} \in \mathcal{F}$ , then either  $u \sim v$ , or at least one of the two vertices is isolated in  $G$ .*

*Proof.* This comes directly from Proposition 2.1.  $\square$

Corollary 2.2 helps us to rebuild the original split graph for a given split graph shelling antimatroid, as shown in the following proposition.

**Proposition 2.3.** *Let  $(V, \mathcal{F})$  be a split graph shelling antimatroid with  $\mathcal{F} \neq 2^V$ , then there is a unique split graph  $G$  such that  $(V, \mathcal{F})$  is the split graph shelling antimatroid defined on  $G$ .*

*Proof.* Suppose we have obtained a graph  $G$  such that  $(V, \mathcal{F})$  is the split graph shelling antimatroid defined on it. Because  $\mathcal{F} \neq 2^V$  and Proposition 2.1, the graph  $G$  must have a non-empty subset of vertices  $S$  such that there exist  $a, b$  in  $N(S)$  with  $a \sim b$ .

If we take an element  $v$  in  $V$  such that  $V \setminus \{u, v\} \in \mathcal{F}$  for all  $u$  in  $V \setminus \{v\}$  (so  $v \notin \{a, b\}$ ), Corollary 2.2 leaves two options. Either the vertex  $v$  is isolated in  $G$  or  $v$  forms an edge with every non isolated vertex in  $G$ . Moreover if this element  $v$  is such that  $\{v\} \in \mathcal{F}$ , the existence of the subset  $S$  in the graph and Proposition 2.1 imply that  $v$  must be an isolated vertex in the graph.

We now build the graph  $G = (V, E)$  as follows. First, we identify the isolated vertices as the vertices  $i$  satisfying  $V \setminus \{i, u\} \in \mathcal{F}$  for all  $u$  in  $V \setminus \{i\}$  and  $\{i\} \in \mathcal{F}$ . Next, among all pairs of non isolated vertices  $\{v_1, v_2\}$ , the ones that give an edge in  $G$  satisfy  $V \setminus \{v_1, v_2\} \in \mathcal{F}$ . We know that there is no other edge by Corollary 2.2.  $\square$

Remark that for the split graph vertex shelling antimatroid  $(V, 2^V)$ , there exist several split graphs such that  $(V, 2^V)$  is the split graph shelling antimatroid defined on it. For instance the complete graph on  $V$ , or the graph  $(V, \emptyset)$ .

We now distinguish two classes of feasible sets for the split graph shelling antimatroids. A feasible set  $F$  is an *i-feasible set* if there exists some vertex  $i$  in  $N(F) \cap I$  (by Corollary 2.1, such an  $i$  is unique). On the other hand, a feasible set  $F$  is a *\*-feasible set* when  $N(F) \subseteq K$ . Figure 5 below illustrates the two classes of feasible sets.

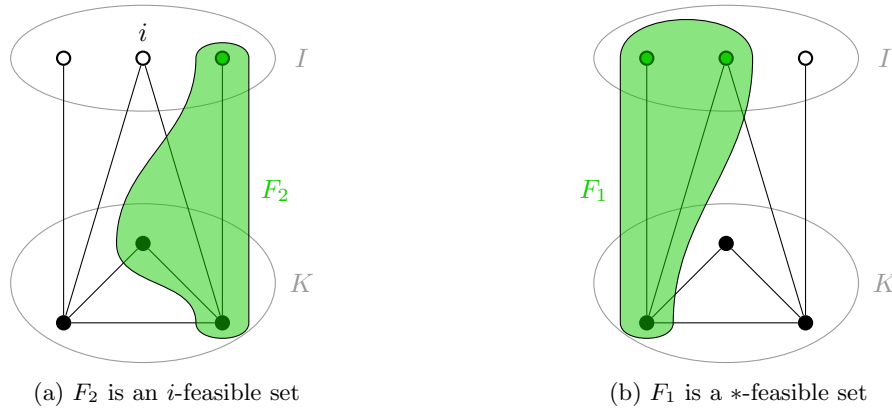


Figure 5: Examples of the two classes of feasible sets

A feasible set of a split graph shelling antimatroid belongs either to the family  $\mathcal{F}_*$  of  $*$ -feasible sets, or to one family  $\mathcal{F}_i$  of  $i$ -feasible sets as shown in the following corollary.

**Corollary 2.3.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ . If  $I = \{i_1, \dots, i_{|I|}\}$ , then  $\mathcal{F}$  decomposes into*

$$\mathcal{F}_* \uplus \mathcal{F}_{i_1} \uplus \dots \uplus \mathcal{F}_{i_{|I|}}.$$

*Proof.* For a feasible set  $F$  there are only two options. If  $N(F) \subseteq K$ , the set  $F$  is a  $*$ -feasible set. And if  $N(F) \not\subseteq K$ , we use Corollary 2.1 to conclude that  $F$  is an  $i$ -feasible set for a specific  $i$  in  $I$ .  $\square$

## Connection between split graph shellings and poset antimatroids

For investigating a split graph  $(K \cup I, E)$ , we will make use of two functions from  $I$  to  $2^{K \cup I}$ , the *forced set function* and the *unforced set function*, respectively:

$$\begin{aligned} \text{fs}(i) &= \{k \in K : k \approx i\} \cup \{i' \in I : N(i') \not\subseteq N(i)\}, \\ \text{uf}(i) &= \text{fs}(i)^c \setminus \{i\} = \{k \in K : k \sim i\} \cup \{i' \in I : N(i') \subseteq N(i)\} \setminus \{i\}. \end{aligned}$$

As shown in the next lemma, the forced set function evaluated at  $i$  gives us the vertices which belong in any  $i$ -feasible set. The unforced set function evaluated at  $i$  just gives the complement of

$\text{fs}(i)$ , minus  $i$ . So for all  $i$  in  $I$  the vertex set of the graph is equal to  $\{i\} \cup \text{fs}(i) \cup \text{uf}(i)$ . Those two definitions are illustrated in Figure 6.

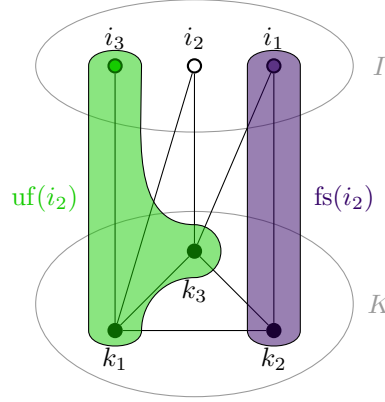


Figure 6: Illustration of the forced set and unforced set functions

**Lemma 2.1.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ . Let  $i$  be in  $I$ , then for any  $i$ -feasible set  $F$  in  $\mathcal{F}$  we have  $\text{fs}(i) \subseteq F$ .*

*Proof.* If  $F$  is an  $i$ -feasible set, we have  $i \in I \cap F^c$  and there is a  $k$  in  $K \cap F$  with  $k \sim i$ . If a vertex  $v$  in  $\text{fs}(i)$  is not in  $F$  then we have two possibilities. Either  $v \in K$  and so  $i \approx v$  (by definition of  $\text{fs}(i)$ ), but that contradicts Proposition 2.1 because  $\{i, v\} \subseteq N(F)$ , or  $v \in I$  and there is  $k' \in K$  such that  $k' \approx i$  but  $k' \sim v$  (by definition of  $\text{fs}(i)$ ). We know that  $k \sim k'$  by definition of  $K$ , but that also contradicts Proposition 2.1 because if  $k' \notin F$ , then  $\{i, k'\} \subseteq N(F)$ , and if  $k' \in F$  then  $\{i, v\} \subseteq N(F)$  with  $i \approx v$  by definition of  $I$ .  $\square$

We will now establish the link between split graph shelling antimatroids and poset antimatroids. If we have a split graph  $G = (K \cup I, E)$ , we build a poset on  $K \cup I$  with the binary relation  $\prec$  defined by  $u \prec v$  if and only if  $u \in K$ ,  $v \in I$  and  $u \sim v$  in  $G$ . The resulting poset  $(K \cup I, \prec)$  is of height at most two (the number of elements in a chain is at most two). Next, we prove that all the structures  $(V, \mathcal{F}_*)$  and  $(\text{uf}(i), \{\text{fs}(i) \cup H : H \in \text{flt}(\text{uf}(i), \prec), H \cap K \neq \emptyset\})$  for  $i$  in  $I$  are poset antimatroids.

**Proposition 2.4.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ , then  $\mathcal{F}_* = \text{flt}(K \cup I, \prec)$ .*

*Proof.* First we show that  $\mathcal{F}_* \subseteq \text{flt}(K \cup I, \prec)$ . Take  $F$  in  $\mathcal{F}_*$ , by definition of a  $*$ -feasible set, if there is a  $k \in F \cap K$ , then  $N(k) \cap I \subseteq F$ . Then  $F$  is a filter of  $(K \cup I, \prec)$ .

Next we show that  $\mathcal{F}_* \supseteq \text{flt}(K \cup I, \prec)$ . Suppose that  $F$  is a filter of  $(K \cup I, \prec)$ , then  $N(F) \subseteq K$  by the definition of  $\prec$ , and by Proposition 2.1 we know that  $F$  is a feasible set because  $K$  is a clique, and also a  $*$ -feasible set because  $N(F) \cap I = \emptyset$ .  $\square$

In the following, we use  $(\text{uf}(i), \prec)$  to denote the poset formed on  $\text{uf}(i)$  with the binary relation  $\prec$  restricted to  $\text{uf}(i)$ .

**Proposition 2.5.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ , then for all  $i$  in  $I$ ,  $\mathcal{F}_i = \{\text{fs}(i) \cup H : H \in \text{flt}(\text{uf}(i), \prec), H \cap K \neq \emptyset\}$ .*

*Proof.* Let  $i$  be in  $I$ . We first show that  $\mathcal{F}_i \subseteq \{\text{fs}(i) \cup H : H \in \text{flt}(\text{uf}(i), \prec), H \cap K \neq \emptyset\}$ . Take a  $F$  in  $\mathcal{F}_i$ , then  $\text{fs}(i) \subseteq F$  by Lemma 2.1. We have directly from the definition of  $\mathcal{F}_i$  that  $(F \setminus \text{fs}(i)) \cap K \neq \emptyset$ , note also that  $F \setminus \text{fs}(i) = F \cap \text{uf}(i)$ . Now we just want to show that  $F \cap \text{uf}(i)$  is a

filter of  $(\text{uf}(i), \prec)$ . This is equivalent to showing that  $N(k) \cap I \cap \text{uf}(i) \subseteq F$  for all  $k$  in  $K \cap (F \cap \text{uf}(i))$ . So if we take a  $k$  in  $K \cap (F \cap \text{uf}(i))$ , then  $k \sim i$ , but  $F$  is a feasible set, so we must have, by the Corollary 2.1,  $N(k) \cap I \cap \text{uf}(i) \subseteq F$ .

Secondly, we show that  $\mathcal{F}_i \supseteq \{\text{fs}(i) \cup H : H \in \text{flt}(\text{uf}(i), \prec), H \cap K \neq \emptyset\}$ . Suppose that  $H$  is a filter of  $(\text{uf}(i), \prec)$  such that  $H \cap K \neq \emptyset$ . We need to show that  $H \cup \text{fs}(i)$  is a feasible set. We use again Proposition 2.1 and check that  $N(\text{fs}(i) \cup H)$  induces a clique. We only need to observe that  $N(\text{fs}(i)) \subseteq N(i)$  by definition of the function  $\text{fs}$ , and this implies  $N(\text{fs}(i) \cup H) \subseteq N(i) \cup \{i\}$  which is a clique. Finally, by construction  $H \cap N(i) \neq \emptyset$  and  $i \notin H$ , so  $\text{fs}(i) \cup H$  is an  $i$ -feasible set and the proof is complete.  $\square$

**Corollary 2.4.** *Let  $G = (K \cup I, E)$  be a split graph and  $(V, \mathcal{F})$  be the split graph vertex shelling antimatroid defined on  $G$ , then  $(V, \mathcal{F}_*)$  and  $(\text{uf}(i), \{F \setminus \text{fs}(i) : F \in \mathcal{F}_i\} \cup \emptyset)$  for  $i$  in  $I$  are all poset antimatroids.*

*Proof.* This follows directly from the Propositions 2.4 and 2.5.  $\square$

Proposition 2.5 shows us that an  $i$ -feasible set can be decomposed into  $\text{fs}(i)$  and a filter of  $(\text{uf}(i), \prec)$ . In fact, this decomposition is unique.

**Proposition 2.6.** *Let  $G = (K \cup I, E)$  be a split graph,  $(V, \mathcal{F})$  the split graph vertex shelling antimatroid defined on  $G$ , and let  $F$  be an  $i$ -feasible set, then  $F = \text{fs}(i) \cup H$ , for exactly one  $H$  in  $\text{flt}(\text{uf}(i), \prec)$ .*

*Proof.* If  $F$  is an  $i$ -feasible set, then  $\text{fs}(i) \subseteq F \subseteq V \setminus \{i\}$ . Moreover, by the definition of  $\text{fs}(i)$ , we have  $V \setminus \{i\} = \text{fs}(i) \cup \text{uf}(i)$ , so the result follows.  $\square$

As said in the introduction, antimatroids have been studied via lattice theory. Indeed, a lattice is created with the feasible sets ordered by inclusion. Then, the lattice operations are:  $F_1 \vee F_2 = F_1 \cup F_2$  and  $F_1 \wedge F_2$  is equal to the maximal feasible set in  $F_1 \cap F_2$ . We call a lattice *join-distributive* if for an element  $x$  in it, the interval  $[x, y]$  is Boolean, where  $y$  is the join of all elements covering  $x$ . We have the following relations between antimatroids and distributive lattices (Edelman [6]).

**Theorem 2.1.** *A finite lattice is join-distributive if and only if it is isomorphic to the lattice of feasible sets of some antimatroid.*

A proof can be found in the book by Korte, Lovász and Schrader [18]. We also have Birkhoff's Representation Theorem [2].

**Theorem 2.2.** *When ordered by inclusion, the feasible sets of a poset antimatroid form a distributive lattice. Conversely, any distributive lattice is isomorphic to some poset antimatroid.*

The above definitions of  $*$ -feasible and  $i$ -feasible sets lead to a better understanding of the lattice produced by the feasible sets of a split graph shelling antimatroid. Indeed, for a split graph shelling antimatroid  $(V, \mathcal{F})$  built on a split graph  $(K \cup I, E)$ , we decompose the structure of the lattice formed by its feasible sets into a distributive lattice  $(\mathcal{F}_*, \subseteq)$  and  $|I|$  distributive lattices  $(\mathcal{F}_i, \subseteq)$ , for  $i \in I$ , as illustrated by Figure 7 and detailed in Example 2.

**Example 2.** Figure 8 shows a split graph  $(K \cup I, E)$  and the lattice formed by the feasible sets of the split graph shelling antimatroid built on it. The dashed link between a feasible set  $F$  and the union sign means that above this point, we look at  $i$ -feasible sets (for some  $i \in I$ ) and the sets considered must be taken in union with  $\text{fs}(i)$ .



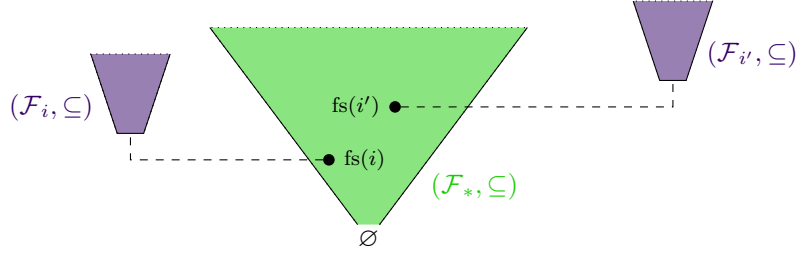


Figure 7: Schematic view of the lattice produced by the feasible sets

### The path poset of a split graph shelling antimatroid

For an antimatroid, we call *path* any feasible set that cannot be decomposed into the union of two other (non-empty) feasible sets. The family of paths can be partially ordered by inclusion, forming the *path poset*. Antimatroids are completely determined by their path posets.

For a split graph shelling antimatroid  $(V, \mathcal{F})$  built on a split graph  $(K \cup I, E)$ , the path poset is easy to obtain in terms of the following sets:

$$\begin{aligned} P_1 &= \{\{i\} : i \in I\}, \\ P_2 &= \{\{k\} \cup (N(k) \cap I) : k \in K\}, \\ P_3 &= \{\text{fs}(i) \cup \{k\} \cup (N(k) \cap I \setminus \{i\}) : i \in I, k \in N(i)\}, \\ P &= P_1 \cup P_2 \cup P_3. \end{aligned}$$

**Proposition 2.7.** *Given a split graph shelling antimatroid  $(V, \mathcal{F})$  built on a split graph  $(K \cup I, E)$  without any vertex  $i$  in  $I$  such that  $N(i) = K$ , its set of paths equals  $P$ .*

In the proposition above, the condition forbidding any  $i$  in  $I$  such that  $N(i) = K$  is not very restrictive because if such an  $i$  exists, we change the partition  $K \cup I$  into  $(K \cup \{i\}) \cup (I \setminus \{i\})$ .

*Proof.* Every set in  $P$  is feasible, because of Proposition 2.1. Next we show that every feasible set in  $P$  cannot be decomposed into the union of two proper feasible sets. This is trivial for the sets in  $P_1$ . So suppose that  $F = \{\{k\} \cup (N(k) \cap I)\}$  in  $P_2$  is the union of two proper sets  $F_1$  and  $F_2$  with  $k \in F_1$ . Then  $F_1$  is not feasible because of Proposition 2.1 and the assumptions which ensures that there is no  $i$  in  $I$  such that  $N(i) = K$ . Now suppose that  $F = \{\text{fs}(i) \cup \{k\} \cup (N(k) \cap I \setminus \{i\})\}$  in  $P_3$  is the union of two proper sets  $F_1$  and  $F_2$  with  $k \in F_1$ . Then  $F_1$  is not feasible because of Proposition 2.1 and  $N(F) \cap I = \{i\}$  and  $N(F) \cap K \subseteq N(i)$ .

Second, we show that every  $F$  in  $\mathcal{F}$  is the union of some sets in  $P$ . If  $F$  is a  $*$ -feasible set, we use the sets from  $P_1$  and  $P_2$ . If  $F$  in an  $i$ -feasible set, we use the sets from  $P_3$  of the form  $\{\text{fs}(i) \cup \{k\} \cup (N(k) \cap I \setminus \{i\})\}$  with  $k \in N(i)$  and some sets from  $P_1$ . By Corollary 2.3 we are done.  $\square$

Figure 9 illustrates Proposition 2.7.

**Corollary 2.5.** *Let  $(V, \mathcal{F})$  be a split graph shelling antimatroid built on a split graph  $(K \cup I, E)$  without any vertex  $i$  in  $I$  such that  $N(i) = K$ . The number of paths in  $(V, \mathcal{F})$  is equals to the number of vertices plus the number of edges from  $K$  to  $I$ .*

*Proof.* This result directly from Proposition 2.7.  $\square$

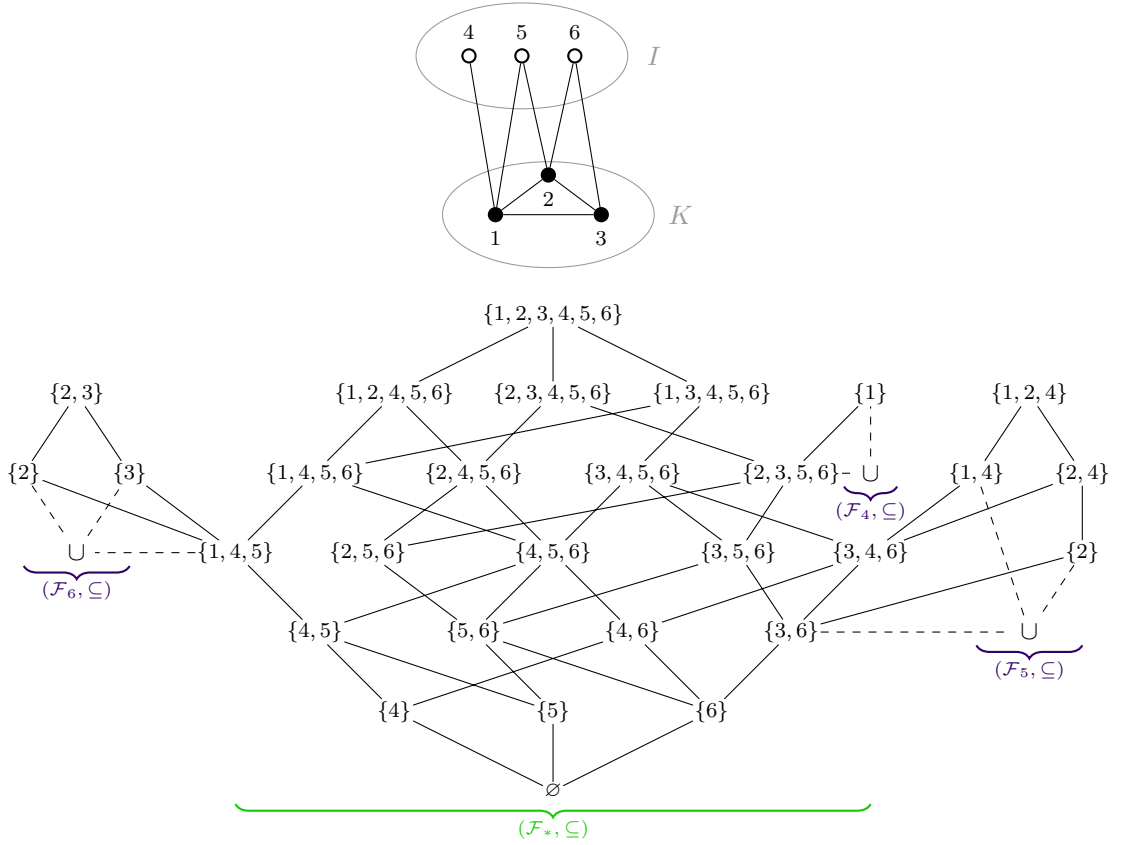


Figure 8: A split graph and the lattices of feasible sets lattices with its shelling antimatroid

### 3 Finding a maximum weight feasible set

#### Hardness result

Many classical problems in combinatorial optimization have the following form. For a set system  $(V, \mathcal{F})$  and for a function  $w : V \rightarrow \mathbb{R}$ , find a set  $F$  of  $\mathcal{F}$  maximizing the value of

$$w(F) = \sum_{f \in F} w(f).$$

For instance, the problem is known to be efficiently solvable for the independent sets of matroids [23] using the greedy algorithm. Since antimatroids capture a combinatorial abstraction of convexity in the same way as matroids capture linear dependence, we investigate the optimization of linear objective functions for antimatroids. It is not known whether a general efficient algorithm exists in the case of antimatroids. Of course, the hardness of finding a maximum weight feasible set depends of the way we encode the antimatroids (see [9] and [7] for more information). If one is given all the feasible sets and a real weight for each element, it is trivial to find a maximum weight feasible set in time polynomial in  $|\mathcal{F}|$  (in the next section we use a better definition of the size of an antimatroid).

We investigate what happens if we chose a more compact way to encode the information. We use now the path poset to describe an antimatroid. However, optimization on antimatroids given in this compact way is hard as Theorem 3.2 shows. We first recall the following theorem due to Håstad [14] initially stated in term of a maximum clique.

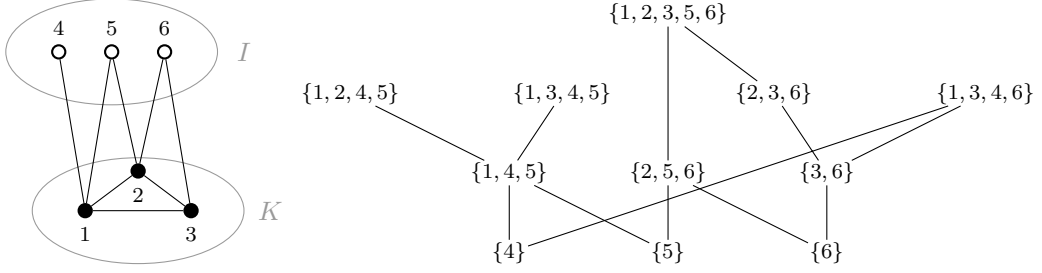


Figure 9: A split graph and the path poset associated with its shelling antimatroid

**Theorem 3.1.** *There can be no polynomial time algorithm that approximates the problem of finding the maximum size of an independent set in a graph on  $n$  vertices to within a factor better than  $O(n^{1-\varepsilon})$ , for any  $\varepsilon > 0$  unless  $\mathcal{P} = \mathcal{NP}$ .*

For antimatroids given in the form of a path poset, there is a  $\mathcal{NP}$ -completeness reduction from the maximum independent set problem to the maximum weight feasible set. The reduction given below is an adaptation of a result due to Eppstein [8].

**Theorem 3.2.** *The problem of finding a maximum weight feasible set in an antimatroid encoded in the form of its path poset is not approximable in polynomial time within a factor better than  $O(N^{\frac{1}{2}-\varepsilon})$  for any  $\varepsilon > 0$ , where  $N$  is the number of elements in the path poset, unless  $P = NP$ .*

*Proof.* Given any graph  $G = (V, E)$  on which we want to find an independent set of maximum size, we define an antimatroid  $(A, \mathcal{F})$  by letting  $A = V \uplus E$  and defining a feasible set (an element of  $\mathcal{F}$ ) as any subset  $F$  of  $A$  such that if  $\{v_1, v_2\} \in E$  and  $\{v_1, v_2\} \subseteq F$  then  $v_1 \in F$  or  $v_2 \in F$ . Remark that  $(A, \mathcal{F})$  is indeed an antimatroid because it satisfies (AM0), (AM1) and (AM2).

The path poset of this antimatroid is composed of sets  $\{v\}$  for each vertex in  $V$  and sets  $\{v, e\}$  for each edge  $e \in E$  such that  $v \in e$ . Let  $d(v)$  denote the degree of the vertex  $v$  and  $\delta = 0.1$ . We define a weight function:  $w : A \rightarrow \mathbb{R}$  by setting

$$w(x) = \begin{cases} -d(x) + \delta & \text{if } x \in V \\ 1 & \text{if } x \in E \end{cases}.$$

We first show that if  $F$  is a feasible set with weight  $w(F)$ , then we can construct an independent set of  $G$  of size at least  $w(F)\delta^{-1}$  in polynomial time. To that end, we define a feasible set  $F' \subseteq F$  as follows. If  $V \cap F$  corresponds to an independent set of vertices in the graph  $G$ , then  $F' = F$ . If it is not the case, we select a pair  $\{u, v\} \subseteq F$  such that  $u \sim v$  in the graph, and remove the element  $u$  from  $F$ . If there is an element  $\{u, a\} \in F \cap E$ , with  $a \notin F$ , we also remove  $\{u, a\}$  from  $F$  (to maintain the feasibility of the set). We repeat this operation until the remaining vertices in the set  $F'$  form an independent set in the graph. The remaining elements then form the set  $F'$ . It is easy to check that  $F'$  is always feasible. By the definition of the function  $w$ , we have the following inequalities,

$$\begin{aligned} w(F) &\leq w(F') \leq \sum_{v \in V \cap F'} (-d(v) + \delta) + \sum_{e \in E \cap F'} 1 \\ &\leq \sum_{v \in V \cap F'} (-d(v) + \delta) + \sum_{v \in V \cap F'} d(v) = \delta |V \cap F'|. \end{aligned}$$

So we have an independent set  $V \cap F'$  that we can construct in polynomial time with size greater

than  $w(F)\delta^{-1}$ .

Now, let  $N$  be the number of paths of  $(A, \mathcal{F})$ , and suppose we have a  $f(N)$ -approximation algorithm to find a maximum weight feasible set *i.e.* we have an algorithm that returns a feasible set with weight at least  $f(N)^{-1}$  times the weight of a maximum weight feasible set. Assume that  $f(N) \leq O(N^{\frac{1}{2}-\varepsilon})$  for some  $0 < \varepsilon < 1$ . We know that  $N = |V| + 2|E|$ , so

$$f(N) \leq O((|V| + 2|E|)^{\frac{1}{2}-\varepsilon}) \leq O((|V| + |V|^2)^{\frac{1}{2}-\varepsilon}) \leq O(|V|^{1-\varepsilon'}),$$

for a  $\varepsilon' \in ]0, 1[$ . So we have

$$\frac{1}{f(N)} \geq \frac{1}{O(n^{1-\varepsilon'})},$$

and we obtain a feasible set with weight at least  $\frac{1}{O(n^{1-\varepsilon'})}$  time the weight  $w^*$  of a maximum weight feasible set. By the previous statement, we build an independent set with size at least  $(w^*/O(n^{1-\varepsilon'}))\delta^{-1}$ , so at least  $1/O(n^{1-\varepsilon'})$  the size of a maximum independent set, and this contradicts Theorem 3.1. So  $f(N) \leq O(N^{\frac{1}{2}-\varepsilon})$  is impossible unless  $\mathcal{P} = \mathcal{NP}$ .  $\square$

Moreover, the above theorem remains true also for a subclass of antimatroids (those built in the proof).

### 3.1 Optimization on split graph shelling antimatroids

We will now prove that for a weighted split graph shelling antimatroid, the problem of finding a maximum weight feasible set can be done in polynomial time in the size of the input even if the form of the input considered is a more compact representation than the path poset. We use the split graph itself to encode all the information about the feasible sets.

In the case of the poset antimatroids, the optimization problem is solved using the solution to the *maximum closure problem*:

**Problem 1.** Given a poset  $(V, \leq)$  and a weight function  $w : V \rightarrow \mathbb{R}$ , find a filter  $F$  that maximizes

$$w(F) = \sum_{f \in F} w(f).$$

Picard [24] designs a polynomial algorithm to solve Problem 1, which calls as a subroutine a maximum flow algorithm (*e.g.* Goldberg and Tarjan [12]). Picard's algorithm runs in  $O(mn \log(\frac{n^2}{m}))$  time, where  $n$  is the number of vertices in  $V$  and  $m$  the number of cover relations in the poset. Taking advantage of this result, we have the following theorem.

**Theorem 3.3.** *Giving a split graph  $G$  (as a list of vertices and a list of edges), the problem of finding a maximum weight feasible set in the split graph shelling antimatroid defined on  $G$  can be done in polynomial time.*

*Proof.* We recall that for every split graph shelling antimatroid we introduce a unique poset with relation  $\prec$  (see just before Proposition 2.4). The construction of this poset combined to Corollary 2.3 and Propositions 2.4 and 2.5 allows us to decompose the problem of finding a maximum feasible set in a split graph antimatroid into several maximum closure problems. Indeed, we first solve the maximum closure problem for  $(V, \prec)$ , yielding a  $*$ -feasible set with maximum weight among all the  $*$ -feasible sets. Then for each  $i$  in  $I$ , we solve the maximum closure problem for  $(\text{uf}(i), \prec)$ , yielding a set  $S$  such that  $S \cup \text{fs}(i)$  is an  $i$ -feasible set that have maximum weight among all  $i$ -feasible sets. The algorithm outputs the feasible set found with maximum weight.  $\square$

So suppose that we have a procedure to find a filter in a poset  $(V, \leq)$  of maximum weight (given by a function  $w$ ) called  $\text{MaxClo}(V, \leq, w)$ . In a split graph  $(K \cup I, E)$ , we look at the element  $i$  in  $I$  that maximize the weight of  $\text{fs}(i) \cup \text{MaxClo}(\text{uf}(i), \prec, w)$ , we then compare the result with the weight of  $\text{MaxClo}(K \cup I, \prec, w)$  and keep the maximum. The time complexity of the algorithm is  $O((|I|(|E||K| + |E||I|) \log(\frac{(|K|+|I|)^2}{|E|})))$  due to the complexity of  $\text{MaxClo}(V, \leq, w)$ . Note that if we use a procedure to find a filter in a poset  $(V, \leq)$  of minimum weight (given by a function  $w$ ), with very little modifications, our algorithm can be used to return a feasible set of minimum weight.

## 4 Free sets and circuits of the split graph shelling antimatroids

In this last section, the term “path” takes only its graph-theoretical meaning, while “circuit” refers to the antimatroidal concept. Our aim is to characterize in simple terms the “circuits” and “free sets” of a split graph shelling antimatroid. Let us first recall some definitions, for a given antimatroid  $(V, \mathcal{F})$ . The *trace* of  $(V, \mathcal{F})$  on a subset  $X$  of  $V$  is

$$\text{Tr}(\mathcal{F}, X) = \{F \cap X : F \in \mathcal{F}\}.$$

A subset  $X$  of  $V$  is *free* if  $\text{Tr}(\mathcal{F}, X) = 2^X$ . A *circuit* is a minimal nonfree subset of  $V$ . An equivalent characterization reads (for a proof, see for instance [17]): a subset  $C$  of  $V$  is a circuit if and only if  $\text{Tr}(\mathcal{F}, C) = 2^C \setminus \{\{r\}\}$  for some  $r$  in  $C$  (this element  $r$  is unique). The element  $r$  is then the *root* of  $C$ , and the pair  $(C \setminus \{r\}, r)$  is a *rooted circuit*. Dietrich [4] shows that the collection of all rooted circuits determines the initial antimatroid, but the collection of circuits themselves does not always share this property. She even shows that the collection of ‘critical’ rooted circuits determines the antimatroid, where a rooted circuit  $(C \setminus \{r\}, r)$  is *critical* when the largest feasible set disjoint from  $C$  is maximal among all the similar feasible sets built for the various circuits rooted at  $r$ . For a recent reference, see Kashiwabara and Nakamura [16].

Let  $G = (V, E)$  be a chordal graph. It is known that the rooted circuits of its vertex shelling antimatroid admits the following simple description: a pair  $(C, r)$  is a rooted circuit if  $C$  consists of two distinct vertices  $u, v$  such that  $r$  is an internal vertex on some chordless path joining  $u$  and  $v$  (this follows immediately from Corollary 3.4 in Farber and Jamison [11]). Moreover, the circuit  $(C, r)$  is critical if and only if the path has exactly three vertices. For the particular case of split graphs we now provide more efficient characterizations of (critical) circuits, and then of free sets.

**Proposition 4.1.** *Let  $(V, \mathcal{F})$  be the vertex shelling antimatroid of the split graph  $(K \cup I, E)$ . Set*

$$\begin{aligned} C_1 &= \{(\{i, j\}, k) : k \in K, i, j \in N(k) \cap I\}; \\ C_2 &= \{(\{i, l\}, k) : k \in K, i \in N(k) \cap I, l \in (N(k) \cap K) \setminus N(i)\}; \\ C_3 &= \{(\{i, j\}, k) : k \in K, i \in N(k) \cap I, j \in I \setminus N(k) \text{ and} \\ &\quad \exists m \in K \text{ with } i \not\sim m, j \sim m\}. \end{aligned}$$

*Then the collection of rooted circuits of  $(V, \mathcal{F})$  equals  $C_1 \cup C_2 \cup C_3$ . Moreover, the collection of critical rooted circuits equals  $C_1 \cup C_2$ .*

*Proof.* Notice that any chordless path in a split graph  $(K \cup I, E)$  has at most four vertices. Moreover, if it has three vertices, the internal vertex is in  $K$  and at least one extremity is in  $I$ . If it has four vertices, the internal vertices are in  $K$  and the extremities in  $I$ . The result then follows from the

characterization of the circuits of the shelling antimatroid of a chordal graph (which we recall just before the statement): the rooted circuits forming  $C_1$  and  $C_2$  come from paths with three vertices, those forming  $C_3$  come from paths with four vertices.  $\square$

**Proposition 4.2.** *Let  $G = (K \cup I, E)$  be a split graph with  $L$  and  $J$  (possibly empty) subsets of respectively  $K$  and  $I$ . Then  $L \cup J$  is free in the vertex shelling antimatroid of  $G$  if and only if either there is no edge between  $L$  and  $J$ , or there exists some vertex  $h$  in  $J$  such that  $L \subseteq N(h)$  and  $N(J \setminus \{h\}) \subseteq N(h) \setminus L$ .*

*Proof.* Assuming first that  $X$  is a free set in  $(V, \mathcal{F})$ , we let  $L = X \cap K$  and  $J = X \cap I$  (we may have  $L$  and/or  $J$  empty). If no edge of  $V$  has an extremity in  $L$  and the other one in  $J$ , then  $L \cup J$  is as in the first case of the statement. If there is some edge  $\{l, h\}$  with  $l \in L$  and  $h \in J$ , we show that  $L$  and  $J$  are as in the second case of the statement. First, there holds  $L \subseteq N(h)$  because otherwise for any vertex  $u$  in  $L \setminus N(h)$ , we would find the circuit  $\{h, l, u\}$  in  $X$  (but a free set cannot contain any circuit). Second, we prove  $N(J \setminus \{h\}) \subseteq N(h) \setminus L$  again by contradiction. Thus assume some vertex  $v$  belongs to  $N(J \setminus \{h\}) \setminus (N(h) \setminus L)$ . Then  $v$  is adjacent to some  $i$  in  $J \setminus \{h\}$ , and  $v$  belongs to either  $L$  or  $K \setminus N(h)$ . In the first eventuality,  $X$  contains the circuit  $\{h, v, i\}$ . In the second eventuality, whether  $l \sim i$  or  $l \not\sim i$ , the circuit  $\{h, l, i\}$  is in  $X$ . In both eventualities we reach a contradiction. Thus  $X = L \cup J$  is as in the second case in the statement.

Conversely, assume  $L$  and  $J$  are as in the statement and let us prove that  $X = L \cup J$  contains no circuit, and so that  $X$  is free. If a rooted circuit  $(\{i, j\}, k)$  from  $C_1$  (as in Proposition 4.1) is in  $X$ , our assumption imposes  $i = h = j$ , a contradiction. If a rooted circuit  $(\{i, l\}, k)$  from  $C_2$  is in  $X$ , then our assumption implies first  $i = h$  because  $l \in N(i) \cap L$ , and then  $k \notin L$  in contradiction with  $k \in X$ . Finally, if a rooted circuit  $(\{i, j\}, k)$  from  $C_3$  is in  $X$  with  $m$  as in  $C_3$ , our assumption implies  $i = h$ , but then  $m \in N(k) \setminus N(h)$  is a contradiction with the assumption.  $\square$

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